TPP Linear Algebra Review
Spring 2021
Lecture Notes

Manon Revel

January 16, 2021

Contents

1 Basic definitions

1.1 Matrices and Vectors

A matrix is an array of numbers organized in rows and columns.

A $\in \mathcal{M}_{n\times m}(\mathbb{R})$ is an (n,m)-matrix. It has n rows and m columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

A vector can be seen as a degenerated matrix — a matrix with only one column. Conversely, a matrix with m columns is formed by m column vectors.

u $\in \mathbb{R}^n$ is a vector.

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

1.2 Operations on Matrices and Vectors

1.2.1 Addition

Let A$\in \mathcal{M}_{n\times m}(\mathbb{R})$ and B$\in \mathcal{M}_{n\times m}(\mathbb{R})$, two matrices with the same dimensions. Then, one can define the function

$$\begin{aligned} \mathcal{M}_{n\times m}(\mathbb{R}) \times \mathcal{M}_{n\times m}(\mathbb{R}) &\longrightarrow \mathcal{M}_{n\times m}(\mathbb{R}) \\ (A, B) &\longrightarrow A + B \end{aligned}$$

where:

$$\mathbf{A+B} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1m}+b_{1m} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2m}+b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+b_{n1} & a_{n2}+b_{n2} & \cdots & a_{nm}+b_{nm} \end{bmatrix}$$

1.2.2 Multiplication by a scalar

Let A$\in \mathcal{M}_{n\times m}(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Then one can define the function

$$\begin{aligned} \mathbb{R} \times \mathcal{M}_{m\times p}(\mathbb{R}) &\longrightarrow \mathcal{M}_{n\times m}(\mathbb{R}) \\ (\lambda, A) &\longrightarrow \lambda \cdot A \end{aligned}$$

where:

$$\lambda\mathbf{A} = \lambda * \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} \lambda \times a_{11} & \lambda \times a_{12} & \cdots & \lambda \times a_{1m} \\ \lambda \times a_{21} & \lambda \times a_{22} & \cdots & \lambda \times a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda \times a_{n1} & \lambda \times a_{n2} & \cdots & \lambda \times a_{nm} \end{bmatrix}$$

1.2.3 Multiplication of matrices

Let A$\in$ $\mathcal{M}_{n\times m}(\mathbb{R})$ and B$\in$ $\mathcal{M}_{m\times p}(\mathbb{R})$ two matrices. Note that the number of columns in A is equal the number of rows in B. Then, the function:

$$\begin{aligned}\mathcal{M}_{n\times m}(\mathbb{R}) \times \mathcal{M}_{m\times p}(\mathbb{R}) &\longrightarrow \mathcal{M}_{n\times p}(\mathbb{R}) \\ (A,B) &\longrightarrow AB\end{aligned}$$

where:

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} * \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^m a_{1k}b_{k1} & \sum_{k=1}^m a_{1k}b_{k2} & \cdots & \sum_{k=1}^m a_{1k}b_{kp} \\ \sum_{k=1}^m a_{2k}b_{k1} & \sum_{k=1}^m a_{2k}b_{k2} & \cdots & \sum_{k=1}^m a_{2k}b_{kp} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^m a_{nk}b_{k1} & \sum_{k=1}^m a_{nk}b_{kp} & \cdots & \sum_{k=1}^m a_{nk}b_{kp} \end{bmatrix}$$

In sum, element (i,j) in AB is $\sum\limits_{k=1}^{m} a_{ik}b_{kj}$.

1.2.4 Transposition

Let A$\in$ $\mathcal{M}_{n\times m}(\mathbb{R})$, then A^T is obtained by switching the rows and columns indices of A. Hence, A$\in$ $\mathcal{M}_{m\times n}(\mathbb{R})$

$$\mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{bmatrix}$$

Note that the following equalities hold:

- $(A^T)^T = A$.
- $(A+B)^T = A^T + B^T$.
- $(\lambda A)^T = \lambda \cdot A^T$
- $(A \times B)^T = B^T \times A^T$.

Exercise: Let $u \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$. Compute $u^T v$ and vu^T. How about uv?

1.2.5 Trace

The trace of a squared matrix is the sum of its diagonal elements. Note that a trace is solely defined for squared matrix.

Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$.

$$Tr(A) = \sum_{i=1}^{n} a_{ii}$$

Example:

$$A = \begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 1 & 1 \\ 7 & 1 & 15 & 12 \\ 6 & 0 & 7 & 0 \end{bmatrix}$$

$Tr(A) = 1 + 1 + 15 + 0 = 17$.

1.3 Some special matrices

1. A *diagonal matrix* is such that $a_{ij} = 0$, $\forall i \neq j$.
2. An *upper-triangular matrix* is such that $a_{ij} = 0$, $\forall i > j$.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nm} \end{bmatrix}$$

3. A *lower-triangular matrix* is such that $a_{ij} = 0$, $\forall i < j$.

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

4. The *identity matrix* is a diagonal matrix where $a_{ii} = 1$, $\forall i$.
5. An *idempotent matrix* is a matrix A such that $A^2 = AA = A$.
6. An *involutive matrix* is a matrix A such that $A^2 = I$.
7. A *symmetric matrix* is a matrix A such that $A = A^T$.
8. An *antisymmetric matrix* is a matrix A such that $A = -A^T$.
9. An *orthonormal matrix* is a matrix A such that $AA^T = I$
10. A *positive semi-definite matrix* is a symmetric matrix such that, $\forall u \in \mathbb{R}^n$, $0 \leq u^T A u$.
11. A *positive definite matrix* is a symmetric matrix such that, $\forall u \in \mathbb{R}^n$, $0 < u^T A u$.

2 Vector space

2.1 Inner Product and Norms

Les E be a set of vectors. If E is closed under addition and multiplication by a scalar, we say that E is a vector space. For $n \in \mathbb{N}$, $\mathbb{R}^n$ is a vector space.

Exercise: Check that $\mathbb{R}^n$ is indeed a vector space.

One can provide the vector space with additional structure, in order to relate two vectors to one another.

An inner-product is a bi-linear (linear in all its components) semi-definite form. It maps two vectors to a scalar.

> Let's define the dot product, an inner-product on $\mathbb{R}^n$:
>
> $$\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$
> $$(u, v) \longrightarrow u^T v = \sum_{i=1}^{n} u_i v_i$$

The following facts hold:

- For $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, $\langle u, v\rangle = \langle v, u\rangle$.
- For $t \in \mathbb{R}^n$ $\langle u, v + t\rangle = \langle u, v\rangle + \langle u, t\rangle$.
- For $\lambda_i \in \mathbb{R}$, $\langle \lambda_1 \cdot u, \lambda_2 \cdot v\rangle = \lambda_1 \lambda_2 \langle u, v\rangle$.
- If $\langle u, v\rangle = 0$, the vectors u and v are orthogonal.

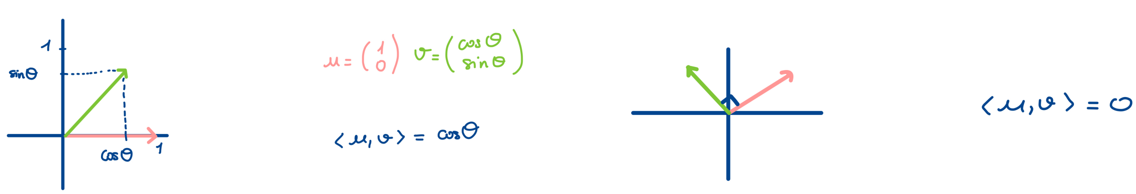


Figure 1: The dot product relates to the angle formed by the vectors.

We can derive a the Euclidean norm from the dot product:

$$||.||_2 : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$u \longrightarrow \sqrt{\langle u, u\rangle} = \sqrt{\sum_{i=1}^{n} u_i^2}$$

The Euclidean norm is an intuitive concept. It is the geometric distance we are used to in 2D. Yet, other norms exist. Formally, a norm is a function that is positive, positive definite (the only vector such that the norm is null is the zero vector), absolutely homogeneous and that satisfies the triangular inequality.

$\forall p$, the following defines a norm:

$$||.||_p : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$u \longrightarrow (\sum_{i=1}^{n} |u_i|^p)^{1/p}$$

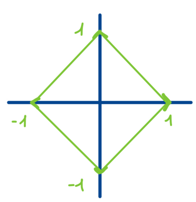
. Further, the infinity norm is defined by:

$$||.||_\infty : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$u \longrightarrow \max_i u_i$$

$(\mathbb{R}^n, ||.||_p)$ is then a normed vector space.

The choice of the norm will "shape" the space's objects. To see this, let us visualize the unit circle in $(\mathbb{R}^2, ||.||_p)$ for $p = 1, 2$ and ∞. The unit circle is a sub-space of $\mathbb{R}^n$ that contains the vectors such that $||.||_p = 1$.

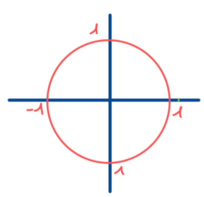
Let $\mathcal{C}_p = \{u \in \mathbb{R}^2 \text{ s.t. } ||u||_p = 1\}$.



$\mathcal{C}_1 = \{ \begin{pmatrix} x \\ y \end{pmatrix} \mid |x| + |y| = 1 \}$

* For $x, y > 0$, $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{C}_1 \Leftrightarrow y = 1 - x$
* For $x < 0, y > 0$ $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{C}_1 \Leftrightarrow y = 1 + x$

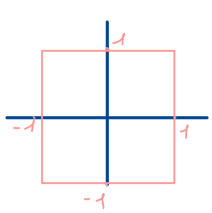
...



$\mathcal{C}_2 = \{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \sqrt{x^2 + y^2} = 1 \}$

* $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{C}_2 \Leftrightarrow x^2 + y^2 = 1$

equation of a circle with unit radius



$\mathcal{C}_\infty = \{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \max(x, y) = 1 \}$

2.2 Family of Vectors and Basis

Let $\mathcal{F} = \{u_1, \cdots, u_p\}$ be a family of vectors in $\mathbb{R}^n$.

$\mathcal{F}$ **spans** a space if and only if any vector of this space can be written as a linear combination of the vectors in the family. That is, the family $\mathcal{F}$ of vectors in space $\mathbb{R}^n$ spans $\mathbb{R}^n$ if and only if:

$$\forall v \in \mathbb{R}^n \ \exists \{c_1, \cdots, c_p\} \text{ s.t. } v = c_1 u_1 + \cdots + c_p u_p$$

.

The vectors in $\mathcal{F}$ **are linearly independent** if and only if no vector in the family can expressed as a linear combination of the others. That is, the vectors in $\mathcal{F}$ are linearly independent if and only if:

$$\forall \{c_1, \cdots, c_p\} \text{ s.t. } c_1 u_1 + \cdots + c_p u_p = [0 \cdots 0]^T \implies c_1 = \cdots = c_p = 0$$

.

A family of vectors $\mathcal{F}$ that is linearly independent and that spans a vector space is a **basis** of this vector space.

Further, if each pair (u_i, u_j) of vector in $\mathcal{F}$ verifies: $\langle u_i, u_j \rangle = 0$, the basis is orthogonal. In addition, if $||u_i|| = 1$ for all i, the basis is orthonormal.

Examples: Let $n = 2$. Let's define

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- The family $\{e_1, e\}$ spans $\mathbb{R}^2$ since we can express any vector in $\mathbb{R}^2$ as a linear combination of e_1 and e. Further, the vectors are linearly independent, since: $c_1 e_1 + c_2 e_2 = 0 \implies [c_1 + c_2, c_2] = [0, 0] \implies c_1 = c_2 = 0$. $\{e_1, e\}$ is a basis for $\mathbb{R}^2$ but it is not an orthogonal basis.
- The family $\{e_1, e_2\}$ spans $\mathbb{R}^2$ and is linearly independent. It is an orthonormal basis.
- The family does not $\mathbb{R}^2$, since the vector e_2, for instance, cannot be expressed as a linear combination of the vectors in the family. The vectors are obviously not linearly independent.
- The family $\{e_1, e_2, e\}$ spans $\mathbb{R}^2$, but it is not linearly independent because $e = e_1 + e_2$.

3 Matrices as linear transformations

3.1 Matrix Rank

Let's go back to our matrices. Let $A \in \mathcal{M}_{n \times m}(\mathbb{R})$.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_m \\ | & | & \cdots & | \end{bmatrix}$$

where vector $v_i \in \mathbb{R}^n$ is A's ith column.

The **rank** of A is the number of linearly independent columns of A. A is full-ranked if rank(A) = n.

Examples:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

- rank(A_1) = 2. A_1 is full-ranked.
- rank(A_2) = 1.
- rank(A_3) = 2.

3.2 Matrix and its Linear Form

Let's work in $\mathbb{R}^2$. Let f be a linear transformation, defined as follow:

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Note that if one defines:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

one can re-write:

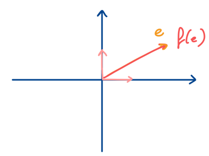
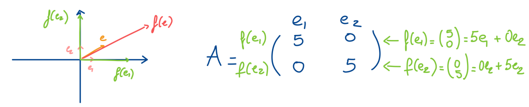
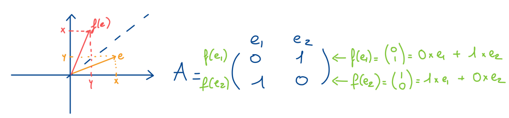
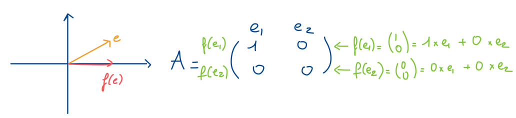
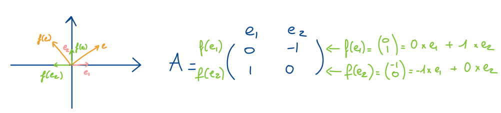
$$f(\begin{bmatrix} x \\ y \end{bmatrix}) = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

f and A are canonically associated. In other words, a matrix A is always associated with a linear transformation.

In fact, A is simply the expression of the projection of f on the canonical basis.

Indeed, let $\mathcal{B} = (e_1, e_2)$ be the canonical base of $\mathbb{R}^2$. $\exists(a, b)$ such that $f(e_1) = ae_1 + be_2$ since $\mathcal{B}$ is a basis. Similarly, $\exists(c, d)$ such that $f(e_2) = ce_1 + de_2$. The matrix A is simply the matrix formed by the coefficients of $f(e_i)$ expressed as a linear combination of e_1, e_2.

Examples:

Type	Linear Function	Plot	Matrix
Identity	$f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x \\ y \end{bmatrix}$		$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Stretching	$f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 5x \\ 5y \end{bmatrix}$		$A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$
Reflection	$f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} y \\ x \end{bmatrix}$		$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Projection	$f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x \\ 0 \end{bmatrix}$		$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Rotation of $\frac{\pi}{2}$	$f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} -y \\ x \end{bmatrix}$		$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Let $E(A) = \{f(x) = Ax \text{ s.t. } x \in \mathbb{R}^2\}$ the space induced by the linear transformation f. The dimension of the induced space is equal to the matrix A's rank.

3.3 Determinant of a Matrix

The determinant of A is related to the volume scaling factor of the linear form canonically associated with A.

- If D is diagonal, $Det(A) = \prod_{i=1}^{n} a_{ii}$.
- If T is triangular, $Det(T) = \prod_{i=1}^{n} a_{ii}$.
- Let $A \in \mathcal{M}_{2\times2}(\mathbb{R})$. Then $Det(A) = |ad - bc|$.

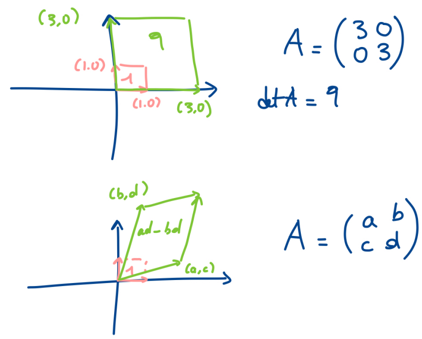


Figure 2: Determinant as a volume

Examples:

- $Det(A_1) = 1$
- $Det(A_2) = 0$
- $Det(A_3) = 0$

4 Matrix Inversion

4.1 Inversion Theorem

Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. $B \in \mathcal{M}_{n\times n}(\mathbb{R})$ is the inverse of A if and only if $AB = BA = I_n$ where I_n is the identity matrix of size n.

If B exists, it is unique, and one writes $B = A^{-1}$.

The following statements are equivalent:

1. A is invertible
2. $Det(A) \neq 0$
3. rank(A) = n

Examples:

- $Rank(A_1) = 2$. Further, $Det(A_1) = 1$. A_1 is invertible.

$$A_1 * A_1^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

. Hence,

$$A_1^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

- $Rank(A_2) = 1$ and $Det(A_2) = 0$. Hence, A_2 is not invertible.

4.2 Properties

- $(A^{-1})^{-1} = A$
- $(A^T)^{-1} = (A^{-1})^T$
- If A and B are invertible, then: $(AB)^{-1} = B^{-1}A^{-1}$
- $Det(A^{-1}) = \frac{1}{Det(A)}$
- Let U be an orthonormal matrix (remember, that is $UU^T = I$). Hence, one sees that U is orthonormal if and only if $U^{-1} = U^T$. Note that all the columns of U are orthonormal.

5 Matrix reduction

As we have seen, a squared matrix A of size n represents a linear transformation in the canonical basis of $\mathbb{R}^n$. We have also seen that one vector space has many different possible bases. Can we find a basis $\mathcal{B}$ in which A is *well-behaved* — that is, in which A is diagonal? If so, we say that A is diagonalizable and

$$A \sim \begin{bmatrix} * & 0 & 0 & \cdots & 0 \\ 0 & * & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & 0 \\ 0 & 0 & 0 & \cdots & * \end{bmatrix} = D$$

More precisely, A is diagonalizable if and only if $\exists P \in \mathcal{M}_{n\times n}(\mathbb{R})$ such that $A = PDP^{-1}$, that is $P^{-1}AP$ is diagonal.

We will explore hereafter the cases where A is diagonalizable and we will identify the diagonal values.

5.1 Eigenvalues and Eigenvectors

Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. λ is an eigenvalue of A if there exists $u \in \mathbb{R}^2$ non-null such that $Au = \lambda u$. Then, u is an eigenvector of A associated with the eigenvalue λ.

$Ax = \lambda u \implies (A - \lambda I_n)x = 0$. Since x is non-null, it implies that $Det(A - \lambda I_n) = 0$ is required.

Exercise: Let $A \in \mathcal{M}_{2\times 2}(\mathbb{R})$. Prove that $Det(A - \lambda I_n) = \lambda^2 - \lambda Tr(A) + Det(A)$.

$Sp(A) = \{\lambda \text{ s.t. } Det(A - \lambda I_n) = 0\}$ is the spectrum of A. It is the set of all eigenvalues of A.

Examples:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

- $$A_1 - \lambda I_2 = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$
  $Det(A_1 - \lambda I_2) = (1-\lambda)^2 = 0 \implies 1 - \lambda = 1$. $Sp(A_1) = \{1\}$.
- $$A_2 - \lambda I_2 = \begin{bmatrix} 2-\lambda & 1 \\ 4 & 2-\lambda \end{bmatrix}$$
  $Det(A_2 - \lambda I_2) = (2-\lambda)^2 - 4 = 0 = -\lambda(4-\lambda)$. $Sp(A_2) = \{0, 4\}$.
- $$A_4 - \lambda I_2 = \begin{bmatrix} 2-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$
  $Det(A_4 - \lambda I_2) = (1-\lambda) \times (2-\lambda) = 0 \implies 1 - \lambda = 1$ or 2. $Sp(A_4) = \{1, 2\}$.

5.2 Eigenspaces and Diagonalization

$\forall \lambda \in Sp(A)$, let $E_\lambda(A) = \{u \text{ s.t. } Au = \lambda u\}$ be the eigenspace associated with eigenvalue λ. It is the space generated by the eigenvectors associated with λ.

Examples:

- Trivially, $Sp(I) = \{1\}$. The vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span $E_1(I)$.
- $Sp(A_1) = \{1\}$.

$$A_1 u = 1 \cdot u \implies \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

  The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ hence spans $E_1(A_1)$.

- $Sp(A_2) = \{0, 4\}$.

$$A_2 u = 0 \cdot u \implies \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ 4x+2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix}$$

  The vector $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ hence spans $E_0(A_2)$.

$$A_2 u = 4 \cdot u \implies \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 4x \\ 4y \end{bmatrix} = \begin{bmatrix} 2x+y \\ 4x+2y \end{bmatrix} = \begin{bmatrix} 4x \\ 4y \end{bmatrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix}$$

  The vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ hence spans $E_4(A_2)$.

- $Sp(A_4) = \{1, 2\}$.

$$A_4 u = 1 \cdot u \implies \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x+y \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix}$$

  The vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ hence spans $E_1(A_4)$.

$$A_4 u = 2 \cdot u \implies \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \begin{bmatrix} 2x+y \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \implies \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}$$

  The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ hence spans $E_2(A_4)$.

Note that vectors that belong to eigenspaces associated with distinct eigenvalues are linearly independent.

A is diagonalizable if and only the eigenvectors of A constitute a basis of $\mathbb{R}^n$.
Equivalently, A is diagonalizable if and only if $\cup_{\lambda \in Sp(A)} E_\lambda(A) = E$.

If such a basis can be found, the vectors in it are used to form the columns of P. Then, $A = PDP^{-1}$. In the basis formed by A's eigenvectors, A is diagonal.

Note that a sufficient condition for A to be diagonalizable is that A has n distinct eigenvalues.

Examples:

- A_1 is not diagonalizable.
- $Sp(A_2) = \{0, 4\}$. A_2 is diagonalizable. $E_0(A) = span(\begin{bmatrix} 1 \\ -2 \end{bmatrix})$ and $E_4(A) = span(\begin{bmatrix} 1 \\ 2 \end{bmatrix})$. $A_2 \sim D = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$. The modal matrix is obtained by concatenating the eigenvectors of A_2 that span the eigenspaces. Hence, $P = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$. One can check that: $P^{-1} = \begin{bmatrix} 1/2 & -1/4 \\ 1/2 & 1/4 \end{bmatrix}$ and that $PDP^{-1} = A_2$.
- $Sp(A_4) = \{1, 2\}$. A_4 is diagonalizable. $E_1(A) = span(\begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and $E_2(A) = span(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$. $A_4 \sim D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. The modal matrix is obtained by concatenating the eigenvectors of A_4 that span the eigenspaces. Hence, $P = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$. One can check that: $P^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ and that $PDP^{-1} = A_4$.

5.3 Specific cases

- Identity matrix has one eigenvalue, 1, that has multiplicity n. It is trivially diagonalizable.
- The reflection matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has two eigenvalues, 1 and -1. It is then diagonalizable.
- The rotation matrix $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has two eigenvalues, i and $-i$. It is not diagonalizable over $\mathbb{R}$.
- **Spectral Theorem:** Let A be a symmetric matrix over $\mathbb{R}$. Then, A's eigenvalues are real and the eigenvectors associated with the eigenvalues are orthogonal and constitute a basis for $\mathbb{R}^n$. Equivalently, there exists an orthonormal matrix U and a diagonal matrix D such that $A = UDU^T$. There exist a basis of orthonormal vectors $\{u_1, \cdots, u_n\}$ such that $A = \sum_{i=1}^{n} \lambda_i u_i u_i^T$ where the λ_i are real.
- A is positive semi-definite if and only if $\forall \lambda \in Sp(A)$, $0 \leq \lambda$.
- A is positive definite if and only if $\forall \lambda \in Sp(A)$, $0 < \lambda$.

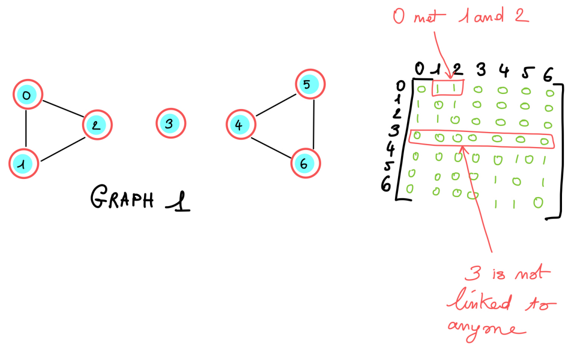
Exercises:

- Prove that two eigenvectors of a symmetric matrix need to be orthogonal.
- Prove that the eigenvalues of a positive-semi definite matrix are non-negative.

6 Application to Networks

Let's see how matrices can help us represent and analyze networks.

Let's represent social ties in a graph as shown below.



6.1 Graph

Let $G = (V, E)$ be a graph with a set of nodes, V, and a set of edges, E. Here, $V = \{0, 1, 2, 3, 4, 5, 6\}$ and $E = \{(0, 1), (0, 2), (1, 2), (4, 5), (4, 5), (5, 6)\}$.

6.2 Adjacency matrix

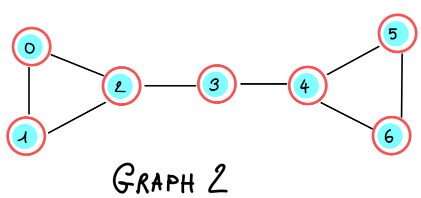
Let $|V| = n$. One can represent G in a matrix form.

Let A be G's adjacency matrix. $A \in \mathcal{M}_{n \times n}(0, 1)$.

$$A_{i,j} = \begin{cases} 1 & \text{if } (i, j) \in E \text{ or if } (j, i) \in E \\ 0 & \text{otherwise} \end{cases}$$

Here, note that $(i, j) \in E \implies (j, i) \in E$. This defines an undirected graph. The adjacency matrix of an undirected graph is symmetric.

Exercise: Write the adjacency matrix of the following graph. Verifies that A is symmetric.



6.3 Node's degree

Let's define the degree of a node $v \in V$ as its number of neighbours.

For $v \in V$:

$$deg(v) = \sum_{i=1}^{n} A_{v,i}$$

The nodes' degree provides information about the nodes' influence, or centrality in the network. Yet, other centrality measures exist — that you will discover in IDS.131. ☺ You'll see how the adjacency matrix together with linear algebra are very handy tools to analyze a graph.