OPTIMAL CONGRESS SIZE

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However small the Republic may be, the Representatives must be raised to a certain number, in order to guard against the *cabals of a few*; and however large it may be, they must be divided to certain number, in order to guard against the *confusion of a multitude*.
In the epistemic world, the optimal congress size is linear.

Seminal work has shown the size of congress ought to be the cube-root of the population's — and it is.

However, all is not lost for small congresses — they can still be legitimate.
THE EPISTEMIC WORLD
THE EPISTEMIC APPROACH

- N agents vote on \( \{0, 1\} \)
- Person i votes according to \( X_i \sim Ber(p_i) \)

- Condorcet trade-off or the power of aggregation of imperfect information: there exists a critical mass such that n agents with \( p_i = .501 \) are better than one experts with \( p = .9999 \)

Condorcet Jury Theorem (1785)

If \( \mathbb{E}[\mathcal{D}] > \frac{1}{2} \), \( \lim_{n \to \infty} \mathbb{P}(\bar{X}_n > \frac{1}{2}) = 1 \)

\[
\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i
\]
LITERATURE
Divide n people in K groups of size n/K

What is K, the number of groups, that maximises the probability that the K q* are right?

If one can sample q* > .5, then K = cn.

Note, q* decreases with K.
EPISTEMIC CONGRESS
MODEL

$n$ agents vote on \{0,1\}

Person $i$ has competence level $p_i \sim \mathcal{D}$, votes $X_i \sim Ber(p_i)$

Sort voters: $p_{(1)} \geq \cdots \geq p_{(n)}$

Let $X_{(1)}, \ldots, X_{(n)}$ be random variables denoting the votes

A congress of size $k$ is the $k$ best experts $X_{(1)}, \ldots, X_{(k)}$

A congress is correct if at least half of the experts vote for 1
What is the optimal congress size?

\[ K^* = \arg \max_{1 \leq k \leq n} \mathbb{P} \left[ \sum_{i=1}^{k} X(i) > k/2 \right] \]
\[ K^* = \arg\max_{1 \leq k \leq n} \mathbb{P} \left[ \sum_{i=1}^{k} X(i) > k/2 \right] \]

Theorem 1

Let \( \mathcal{D} = \mathcal{U}[0,1] \), and suppose the competence levels are deterministically their expectation, so \( p(i) = 1 - \frac{i}{n+1} \). Then:

\[ \frac{1}{6} < \frac{K^*}{n} < \frac{1}{2} \]
Theorem 2

Let $\mathcal{D}$ be a distribution supported by $[l, h]$ with $l > 0$ and $h < 1$ whose inverse CDF is Lipschitz continuous. Then, w.p. $\geq 1 - \frac{1}{n}$:

$$1 - F \left( \frac{1}{1 + \sqrt{\frac{1 - H}{H}}} + o(1) \right) - o(1) < \frac{K^*}{n} < 1 - F \left( \frac{1}{1 + \sqrt{\frac{1 - L}{L}}} - o(1) \right) - o(1)$$
PROOF SKETCH

➤ The proofs rely on observing that for an odd $k$ too small, one can always add two experts and increase the accuracy of the congress.

➤ The two added experts are only relevant to the outcome when:

  ➤ exactly $\frac{k+1}{2}$ out of the $k$ initial experts are correct, so adding two incorrect experts reverses the majority decision from correct to incorrect

  ➤ exactly $\frac{k-1}{2}$ out of the $k$ initial experts are correct, so adding two correct experts reverses the majority decision from incorrect to correct

➤ Then, compare the probabilities of these two scenarios.
MORE REALISTIC SAMPLING METHODS

\[ K_S^* = \arg\max_{1 \leq k \leq n} \mathbb{P} \left[ \sum_{i \in S_k} X(i) > k/2 \right] \]

Conjecture

Let's S be a sampling method that defines a sequence \( S_1, \ldots, S_n \) of congresses of increasing size, that allows to sample any agent \( i \) with \( p(i) > \frac{1}{2} \), then \( K_S^* > K^* \).
In the epistemic world, even when the top experts' competence approaches 1, the optimal congress size is linear.

This holds when giving the congress its best shot.
CUBE ROOT LAW AND REAL CONGRESSES
REAL-WORLD CONGRESSES

Log-log plot of the Congress size as a function of the population size and regression line

- Regression line
- Legislatures

United States
Real-world congresses are of size $\approx n^{0.36}$
AN ACTIVE DEBATE

➤ Italian Referendum 2020: Should the congress size be reduced?

- House: 630 to 400
- Senate: 315 to 200
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Demografia e riduzione dei parlamentari. La “legge” della radice cubica

by Giuseppe de Bartolo — 07/16/2019 in Società e Demografia

IL FOGLIO

parlamentari e dimensione della popolazione (parere 662/2012 del 18 giugno 2012). Dunque, a parte il fatto che in concreto, mano a mano che cresce la popolazione, il rapporto tende ad alzarsi (per ovvie ragioni), ogni paese fa come vuole. In ogni caso rimane la sproporzionata attuale del numero dei parlamentari italiani (e il costo delle due Camere doppione che pesano più di qualsiasi altro Parlamento in Europa, sfiorando quello del Congresso Usa) spiega sia il favore di tutte le forze politiche per la riduzione sia la generalizzata opinione popolare nella stessa direzione (entrambe, come si è visto, indipendenti e precedenti le spinte demagogiche di partiti come il M5S).

Segnalò infine una curiosità, per quel che può valere: un importante politologo estone, Rein Taagepera, ha teorizzato che il numero di rappresentanti “ideale” sarebbe la radice cubica della popolazione rappresentata: ebbene applicando la sua “legge”, i 60.400.000 italiani dovrebbero esprimere esattamente 392,14 deputati, un numero straordinariamente vicino ai 392 (più gli 8 che rappresentano i circa quattro milioni di italiani all’estero) che effettivamente essi eleggeranno se la riforma verrà confermata.
LEGITIMATE CONGRESS
CAN SMALL CONGRESSES STILL BE LEGITIMATE?

> Under which conditions is a sub-optimally sized congress still better than majority? That is, under which conditions do we have:

\[
\Gamma(k) = \mathbb{P}\left[ \sum_{i=1}^{k} X(i) > k/2 \right] - \mathbb{P}\left[ \sum_{i=1}^{n} X(i) > n/2 \right] > 0?
\]
Theorem 3

Let $\mathcal{D} = \mathcal{U}[\varepsilon_n, 1]$.

- If $\varepsilon_n = o \left(\sqrt{\frac{\log n}{n}}\right)$, then $\Gamma(1) > 0$.
- If $\varepsilon_n = \omega \left(\sqrt{\frac{\log n}{n}}\right)$, then $\Gamma(1) < 0$. 
Gain $\Gamma(k)$ of Congress over Majority for different biases in society...
...assuming uniform expertise levels on $[\varepsilon_n, 1]$. 

Societal bias towards the truth:
- $\varepsilon_n = \frac{1}{n}$
- $\varepsilon_n = \sqrt{\frac{1}{n}}$
- $\varepsilon_n = \sqrt[4]{\frac{\log \log n}{n}}$
- $\varepsilon_n = \sqrt[4]{\frac{\log n}{n}}$
...assuming uniform expertise levels on $[0.4 + \varepsilon_n, 0.6]$
There exists a critical population size for large biases.

For population sizes too small in comparison to the society biases, majority beats a cube-root congress.

...assuming uniform expertise levels on $[0.4 + \varepsilon_n, 0.6]$. 
CONCLUSION

* Epistemic congresses ought to grow *linearly* with the population's size.

* Yet, other desiderata should be considered (e.g. costs, representativity, practicality).

* *Cube-root* sized congresses can still beat majority under mild conditions.
REAL-WORLD CONGRESSES

Congress size in 240 legislatures

- Real-world Congresses Size across 240 legislatures

E. Auriol and R. J Gary-Bobo. The more the merrier? choosing the optimal number of representatives in modern democracies, 2008


