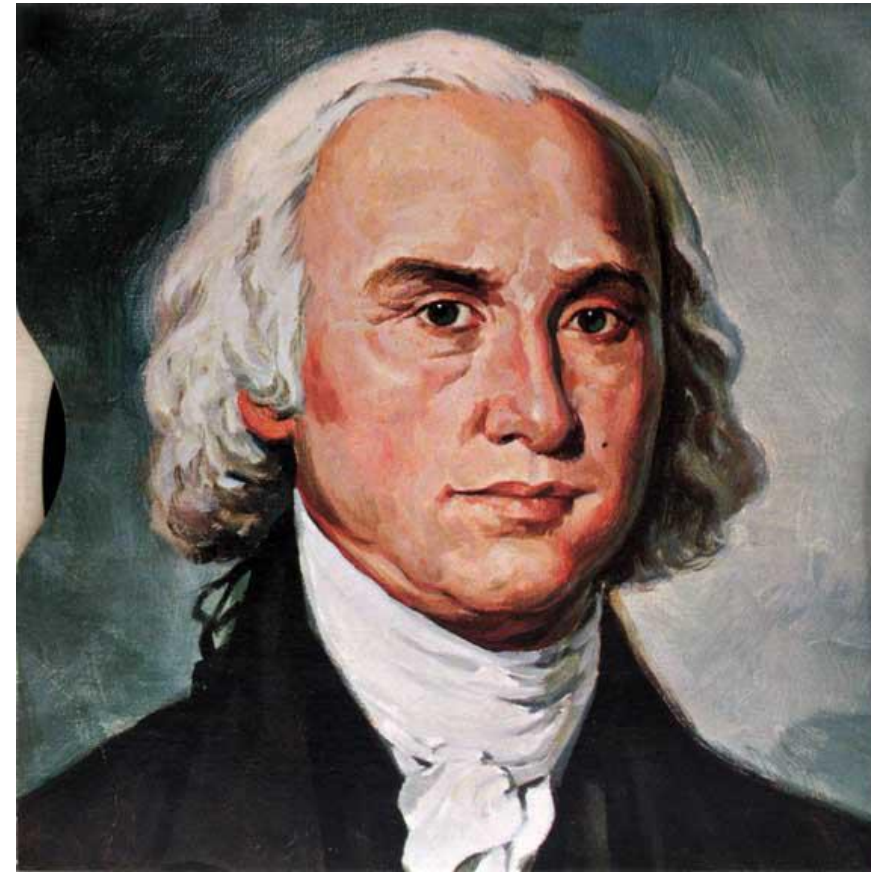


OPTIMAL CONGRESS SIZE

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Group Meeting May 7



James Madison
Fourth president of the United States.

However small the Republic may be, the Representatives must be raised to a certain number, in order to guard against the *cabals of a few*; and however large it may be, they must be divided to certain number, in order to guard against the *confusion of a multitude*.

AGENDA

- ❖ In the epistemic world, the optimal congress size is *linear*.
- ❖ Seminal work has shown the size of congress ought to be the *cube-root* of the population's — and it is.
- ❖ However, all is not lost for small congresses — they can still be *legitimate*.

THE EPISTEMIC WORLD

THE EPISTEMIC APPROACH

- N agents vote on $\{0,1\}$ *ground truth*
- Person i votes according to $X_i \sim \text{Ber}(p_i)$ *where $p_i \sim \mathcal{D}$*
- Condorcet trade-off or the **power of aggregation of imperfect information**: there exists a critical mass such that n agents with $p_i = .501$ are better than one experts with $p = .9999$

✻ Condorcet Jury Theorem (1785)

If $\mathbb{E}[\mathcal{D}] > \frac{1}{2}$,

$$\lim_{n \rightarrow \infty} \underbrace{\mathbb{P}(\bar{X}_n > \frac{1}{2})}_{\text{more than } n/2 \text{ voters got it right}} = 1$$

more than $n/2$
voters got it right

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i$$

LITERATURE



- R. Taagepera. ***The size of national assemblies***. Social science research, 1972.
- E. Auriol and R. J Gary-Bobo. ***The more the merrier? choosing the optimal number of representatives in modern democracies***, 2008
- E. Auriol and R. J Gary-Bobo. ***On the optimal number of representatives***. Public Choice, 2012.
- G. Margaritondo. ***Size of national assemblies: The classic derivation of the cube-root law is conceptually awed***. Frontiers in Physics, 2021.

$$\sqrt[3]{n}$$

$$\sqrt{n}$$

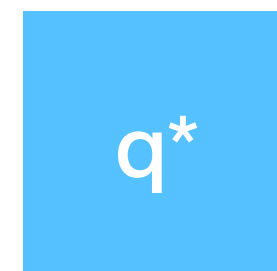
- L. Zhao & T. Peng. ***An allometric scaling for the number of representative nodes in social networks***. Conference on Network Science, 2020.

$$n^{5/9}$$

- M. Magdon-Ismail & L. Lia. ***A Mathematical Model For Optimal Decisions In A Representative Democracy***, NeurIPS, 2018.

$$cn$$

Divide n people in K groups of size n/K



What is K , the number of groups, that maximises the probability that the K q^* are right?

If one can sample $q^* > .5$, then $K = cn$.

Note, q^* decreases with K .

EPISTEMIC CONGRESS

MODEL

..... *ground truth*

- n agents vote on $\{0,1\}$
- Person i has competence level $p_i \sim \mathcal{D}$, votes $X_i \sim \text{Ber}(p_i)$
- Sort voters: $p_{(1)} \geq \dots \geq p_{(n)}$
- Let $X_{(1)}, \dots, X_{(n)}$ be random variables denoting the votes
- A **congress of size k** is the k best experts $X_{(1)}, \dots, X_{(k)}$
- A congress is correct if at least half of the experts vote for 1

WHAT IS THE OPTIMAL CONGRESS SIZE?

- What is the optimal congress size?

$$K^{\star} = \operatorname{argmax}_{1 \leq k \leq n} \mathbb{P} \left[\sum_{i=1}^k X_{(i)} > k/2 \right]$$

UNIFORM[0, 1]

$$K^{\star} = \operatorname{argmax}_{1 \leq k \leq n} \mathbb{P} \left[\sum_{i=1}^k X_{(i)} > k/2 \right]$$

☀ Theorem 1

Let $\mathcal{D} = \mathcal{U}[0,1]$, and suppose the competence levels are deterministically their expectation, so $p_{(i)} = 1 - \frac{i}{n+1}$. Then:

$$\frac{1}{6} < \frac{K^{\star}}{n} < \frac{1}{2}$$

DISTRIBUTIONS BOUNDED AWAY FROM 1

✱ Theorem 2

Let \mathcal{D} be a distribution supported by $[l, h]$ with $l > 0$ and $h < 1$ whose inverse CDF is Lipschitz continuous. Then, w.p. $\geq 1 - \frac{1}{n}$:

$$1 - F\left(\frac{1}{1 + \sqrt{\frac{1-H}{H}}} + o(1)\right) - o(1) < \frac{K^\star}{n} < 1 - F\left(\frac{1}{1 + \sqrt{\frac{1-L}{L}}} - o(1)\right) - o(1)$$

PROOF SKETCH

- The proofs rely on observing that for an odd k *too small*, one can always add two experts and increase the accuracy of the congress.
- The two added experts are only relevant to the outcome when:
 - exactly $\frac{k+1}{2}$ out of the k initial experts are *correct*, so adding two *incorrect* experts reverses the majority decision from *correct* to *incorrect*
 - exactly $\frac{k-1}{2}$ out of the k initial experts are *correct*, so adding two *correct* experts reverses the majority decision from *incorrect* to *correct*
- Then, compare the probabilities of these two scenarios.

MORE REALISTIC SAMPLING METHODS

$$K_S^\star = \operatorname{argmax}_{1 \leq k \leq n} \mathbb{P} \left[\sum_{i \in S_k} X_{(i)} > k/2 \right]$$

✶ Conjecture

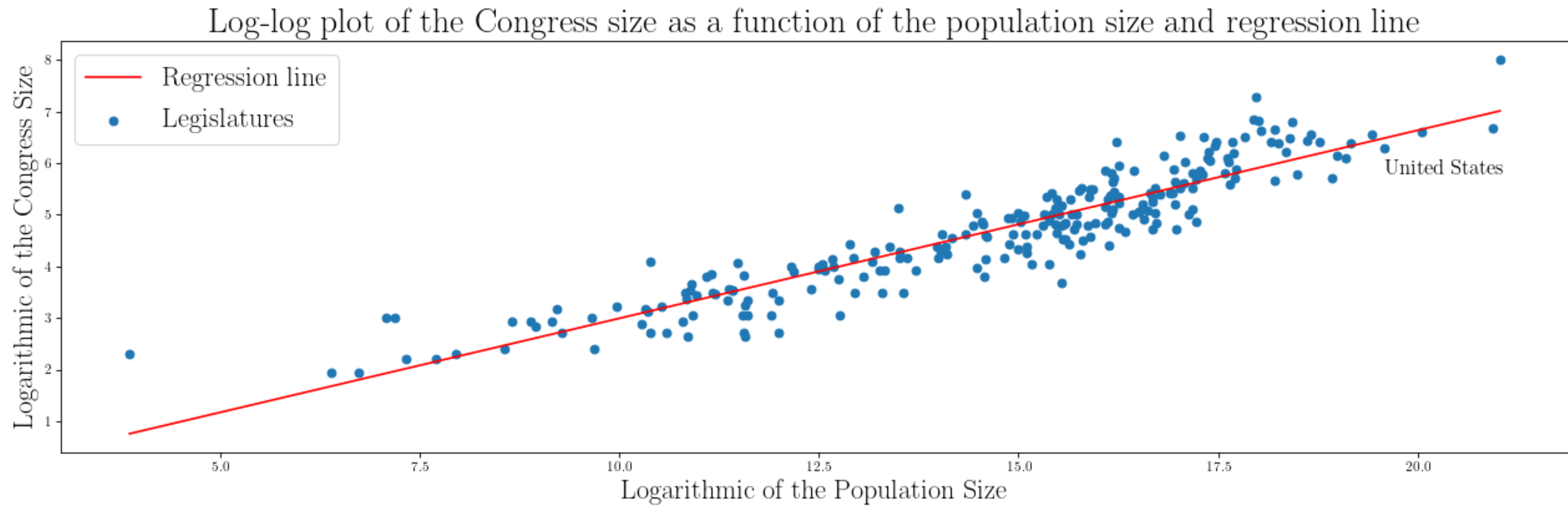
Let's S be a sampling method that defines a sequence S_1, \dots, S_n of congresses of increasing size, that allows to sample any agent i with $p_{(i)} > \frac{1}{2}$, then $K_S^\star > K^\star$.

TAKE AWAY

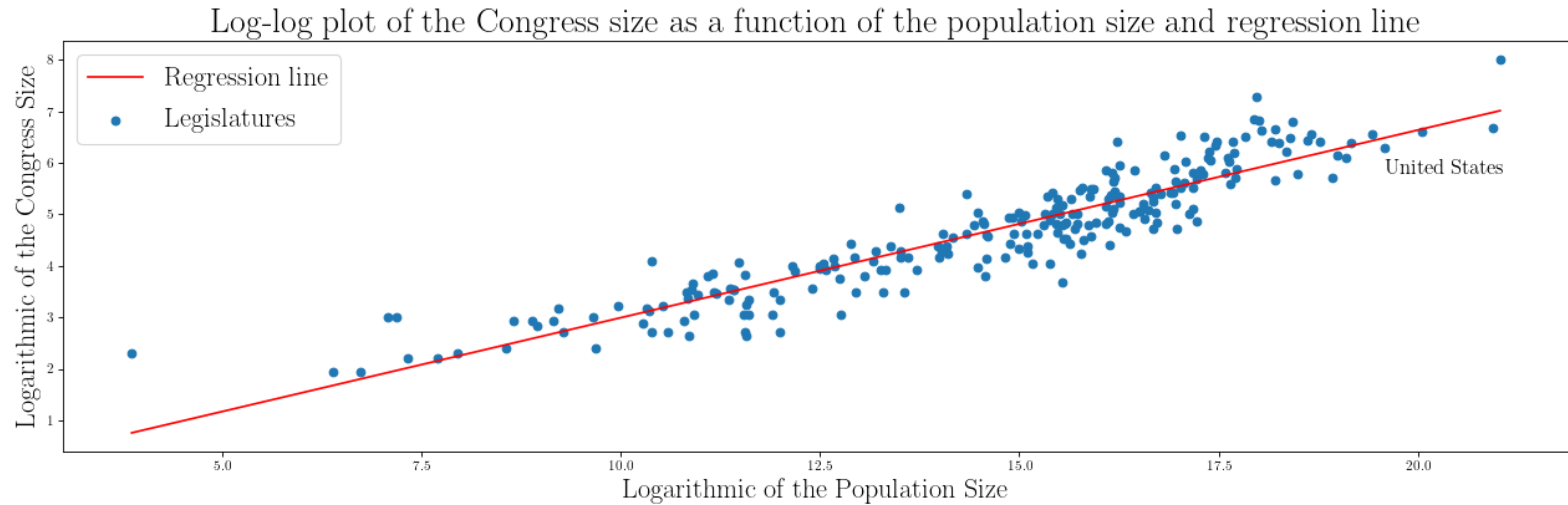
- ❖ In the epistemic world, even when the top experts' competence approaches 1, the optimal congress size is *linear*.
- ❖ This holds when giving the congress its best shot.

CUBE ROOT LAW AND REAL CONGRESSES

REAL-WORLD CONGRESSES



REAL-WORLD CONGRESSES



❖ Real-world congresses are of size $\approx n^{0.36}$

AN ACTIVE DEBATE

- Italian Referendum 2020: Should the congress size be reduced?
 - House: 630 to 400
 - Senate: 315 to 200

AN ACTIVE DEBATE

➤ Italian Referendum 2020: Should the congress size be reduced?

- House: 630 to 400
- Senate: 315 to 200



IL FOGLIO

quotidiano

parlamentari e dimensione della popolazione" (parere 662/2012 del 18 giugno 2012). Dunque, a parte il fatto che in concreto, mano a mano che cresce la popolazione, il rapporto tende ad alzarsi (per ovvie ragioni), ogni paese fa come vuole. In ogni caso **rimane la sproporzione attuale del numero dei parlamentari italiani** (e il costo delle due Camere doppione che pesano più di qualsiasi altro Parlamento in Europa, sfiorando quello del Congresso Usa) spiega sia il favore di tutte le forze politiche per la riduzione sia la generalizzata opinione popolare nella stessa direzione (entrambe, come si è visto, indipendenti e precedenti le spinte demagogiche di partiti come il M5S).

Segnalo infine una curiosità, per quel che può valere: un importante politologo estone, **Rein Tagepera**, ha teorizzato che il numero di rappresentanti “ideale” sarebbe **la radice cubica della popolazione rappresentata**: ebbene applicando la sua “legge”, i 60.400.000 italiani dovrebbero esprimere esattamente 392,14 deputati, un numero straordinariamente vicino ai 392 (più gli 8 che rappresentano i circa quattro milioni di italiani all'estero) che effettivamente essi eleggeranno se la riforma verrà confermata.

$$\sqrt[3]{n}$$



Taagepera 72
Auriol & Gary-Bobo 07-12
Zhao & Peng 20
Margaritondo 21

LEGITIMATE CONGRESS

CAN SMALL CONGRESSES STILL BE LEGITIMATE?

- Under which conditions is a sub-optimally sized congress still better than majority? That is, under which conditions on \mathcal{D} do we have:

$$\Gamma(k) = \mathbb{P} \left[\sum_{i=1}^k X_{(i)} > k/2 \right] - \mathbb{P} \left[\sum_{i=1}^n X_{(i)} > n/2 \right] > 0 ?$$

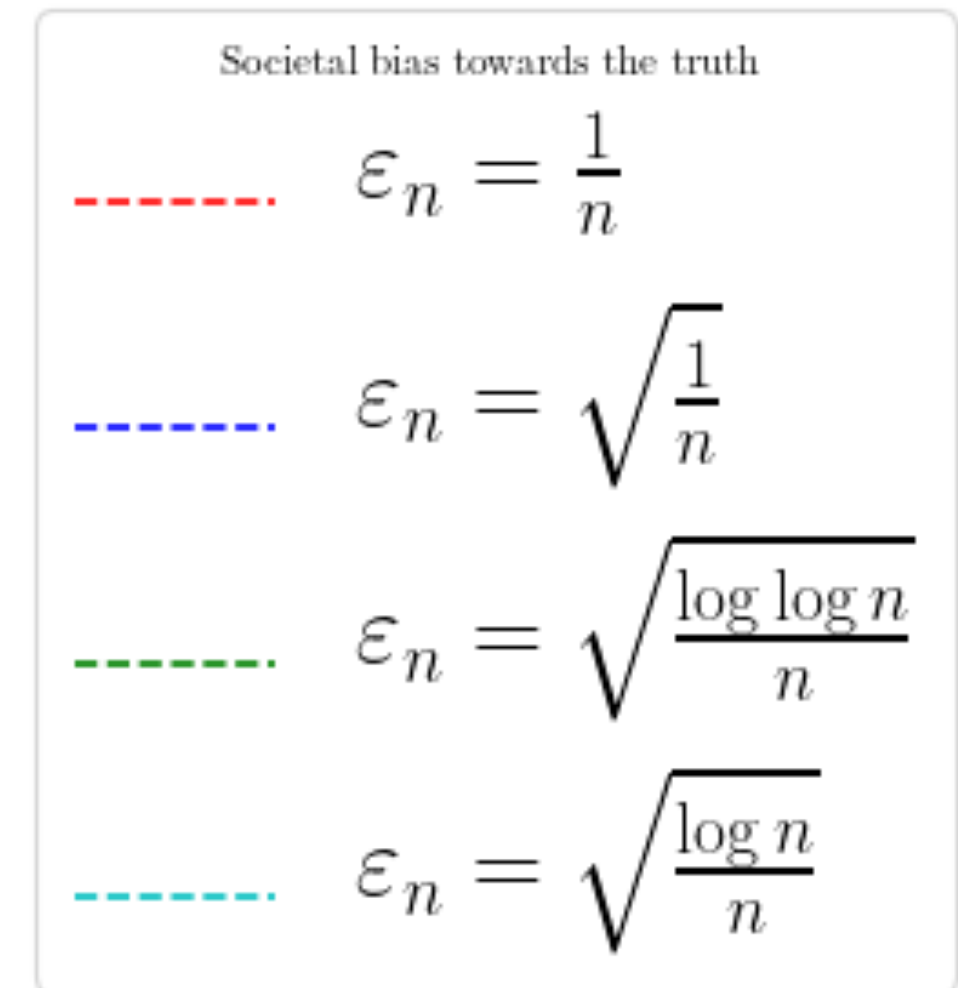
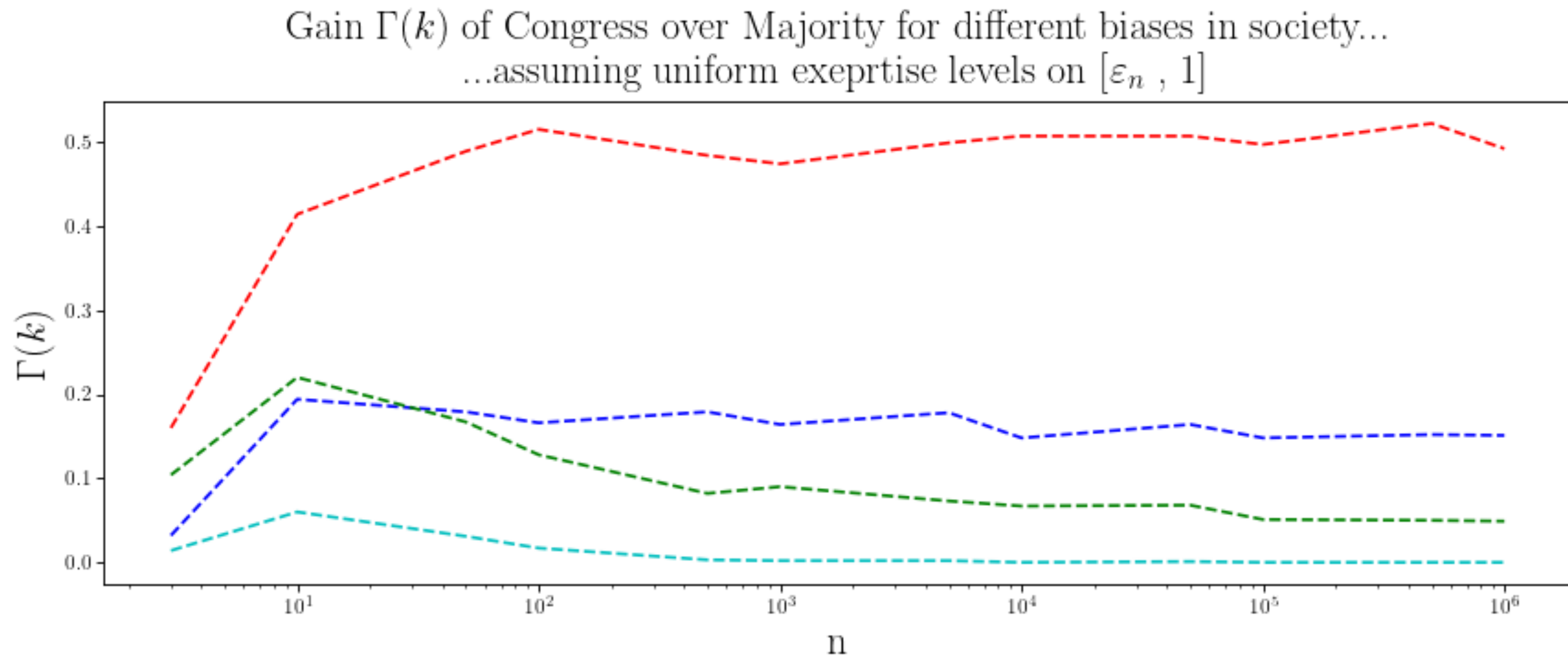
DICTATORSHIP

✱ Theorem 3

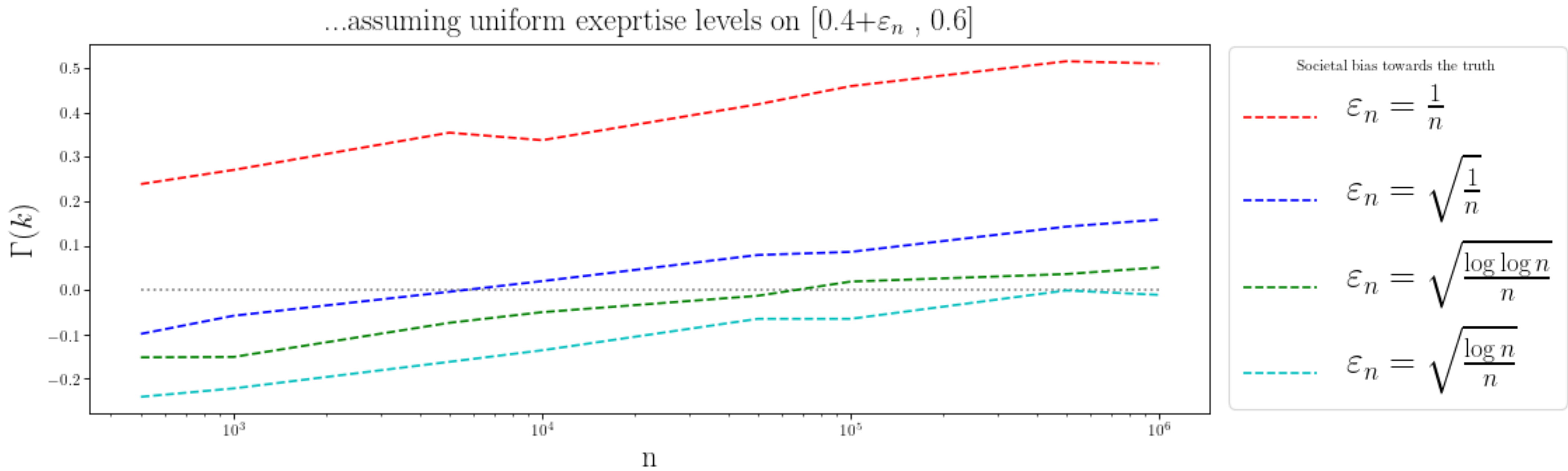
Let $\mathcal{D} = \mathcal{U}[\varepsilon_n, 1]$.

- If $\varepsilon_n = o\left(\sqrt{\frac{\log n}{n}}\right)$, then $\Gamma(1) > 0$.
- If $\varepsilon_n = \omega\left(\sqrt{\frac{\log n}{n}}\right)$, then $\Gamma(1) < 0$.

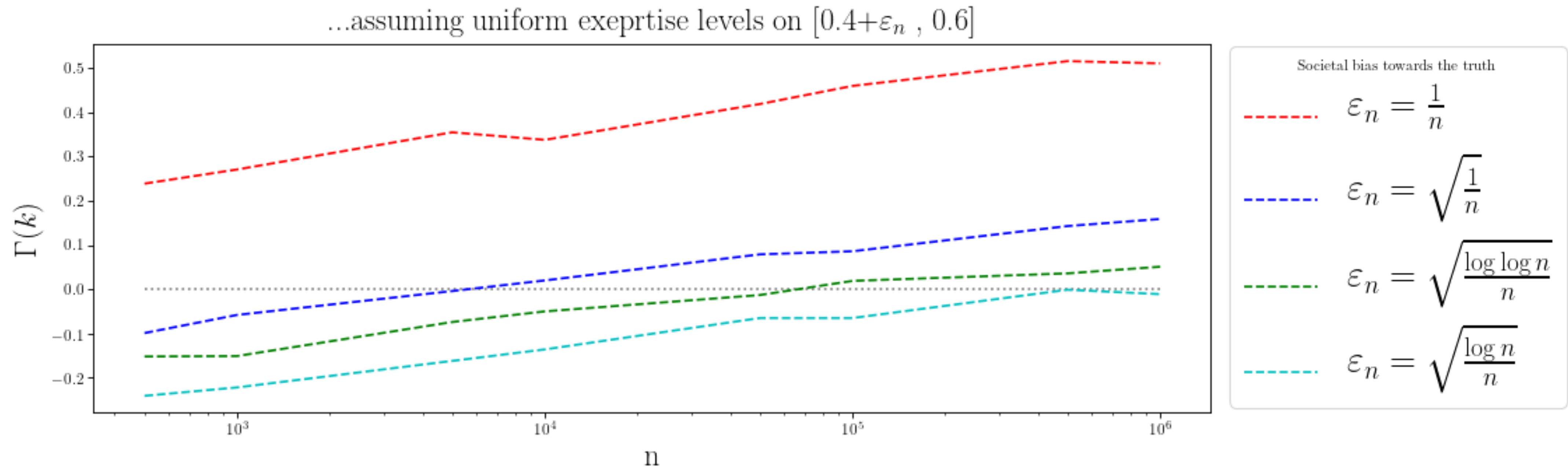
CUBE-ROOT CONGRESS



CUBE-ROOT CONGRESS



CUBE-ROOT CONGRESS

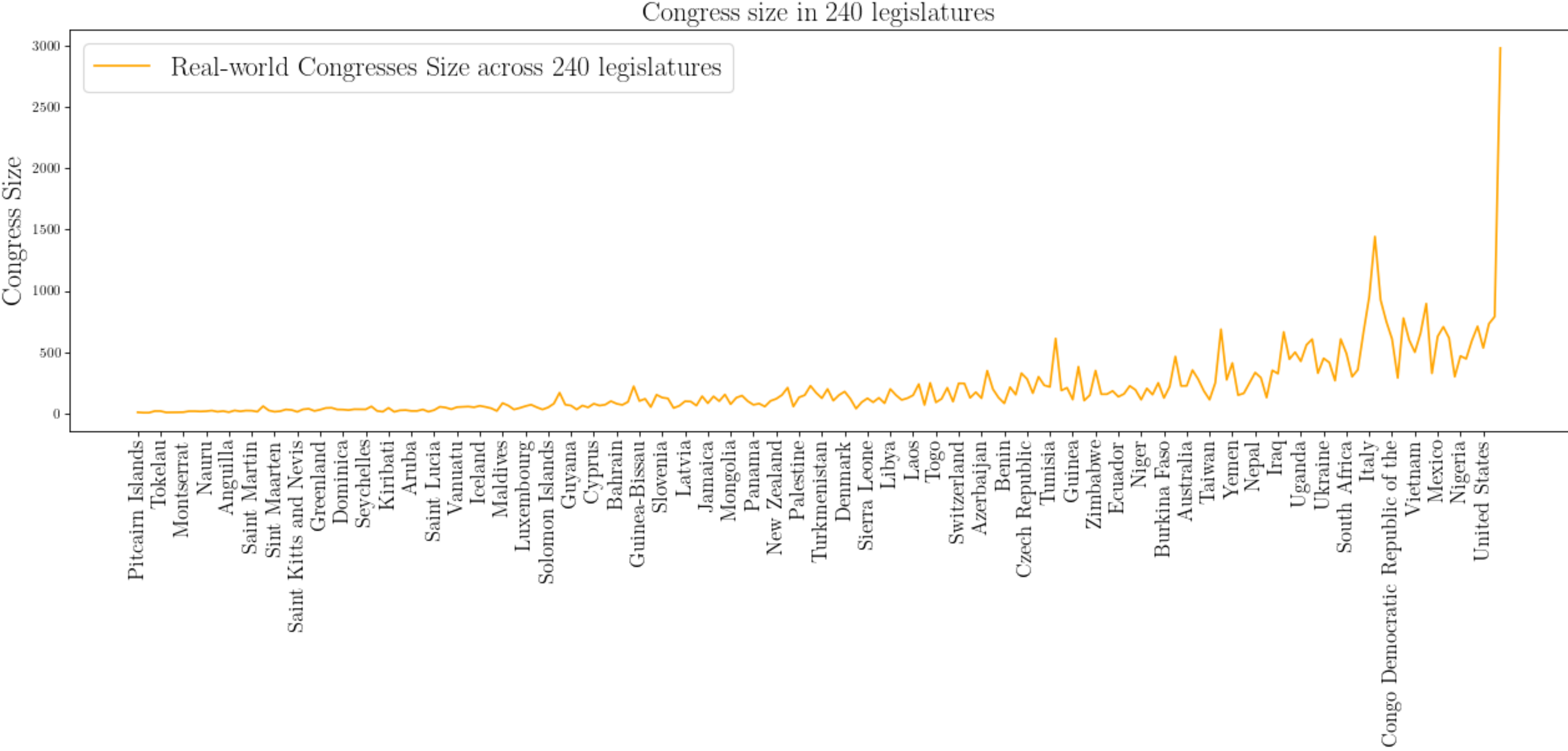


- ❖ There exists a critical population size for large biases.
- ❖ For population sizes too small in comparison to the society biases, majority beats a cube-root congress.

CONCLUSION

- ❖ Epistemic congresses ought to grow *linearly* with the population's size.
- ❖ Yet, other desiderata should be considered (e.g. costs, representativity, practicality).
- ❖ *Cube-root* sized congresses can still beat majority under mild conditions.

REAL-WORLD CONGRESSES





- E. Auriol and R. J Gary-Bobo. ***On the optimal number of representatives***. Public Choice, 2012.
- E. Auriol and R. J Gary-Bobo. ***The more the merrier? choosing the optimal number of representatives in modern democracies***, 2008
- C. Blum and C. I Zuber. ***Liquid democracy: Potentials, problems, and perspectives***. Journal of Political Philosophy, 2016.
- F. Brandt, V. Conitzer, U. Endress, J. Lang, & A. D. Procaccia, editors. ***Handbook of Computational Social Choice***. Cambridge University Press, 2016.
- H. Cramer. ***Sur un nouveau théorème-limite de la théorie des probabilités***. Actual. Sci. Ind, 1932.
- L. De Sio and D. Angelucci. ***945 sono troppi? 600 sono pochi? Qual e il numero ottimale di parlamentari?*** Cise, 2018.
- B. Grofman, G. Owen, and Scott L Feld. ***Thirteen theorems in search of the truth***. Theory and decision, 1983.
- K. Jacobs & S. Otjes. ***Explaining the size of assemblies. a longitudinal analysis of the design and reform of assembly sizes in democracies around the world***. Electoral Studies, 2015.
- A. Kahng, S. Mackenzie, & A. D Procaccia. ***Liquid democracy: An algorithmic perspective***. In 32nd AAAI Conference on AI, 2018.
- G. Margaritondo. ***Size of national assemblies: The classic derivation of the cube-root law is conceptually awed***. Frontiers in Physics, 2021.
- J. C Miller. ***A program for direct and proxy voting in the legislative process***. Public choice, 1969.
- G. Szpiro. ***Numbers rule: the vexing mathematics of democracy, from Plato to the present***. Princeton University Press, 2010.
- R. Taagepera. ***The size of national assemblies***. Social science research, 1972.
- L. Zhao & T. Peng. ***An allometric scaling for the number of representative nodes in social networks***. Conference on Network Science, 2020.